Theorem (Extension operator in 1d)

1. Let $1 \leq p \leq \infty$ and $I \subseteq \mathbb{R}$ be open interval ($I = (a,b)$, $-\infty \leq a < b \leq +\infty$).

Then there exists bounded linear operator $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ s.t.

(i) $Pu|_I = u \quad \forall u \in W^{1,p}(I)$

(ii) $\|Pu\|_{L^p(\mathbb{R})} \leq C \|u\|_{L^p(\mathbb{R})}$

(iii) $\|Pu\|_{W^{1,p}(\mathbb{R})} \leq C \|u\|_{W^{1,p}(I)}$

$C = C(I) > 0$.

<u>Note</u>: wlog we can consider $I = (0,\infty)$, $I = (0,1)$.

(a) $I = (0,\infty)$.

<u>Claim</u>: $(Pu)(x) := \begin{cases} u(x) & x \geq 0 \\ u(-x) & x < 0 \end{cases}$ satisfies (i)-(iii).

<u>Pf</u>: Write $u^* = Pu$.

(i) Clearly $u^*(x) = u(x)$ when $x \in (0,\infty)$ by def

(ii) Clearly $\int_{\mathbb{R}} |u^*|^p \, dx = 2\int_0^\infty |u|^p \, dx < \infty$

so $\|Pu\|_{L^p(\mathbb{R})} = 2^{1/p} \|u\|_{L^p(I)} \quad (\leq 2\|u\|_{L^p(I)})$

(iii) Let $v(x) = \begin{cases} u'(x) & x \geq 0 \\ -u'(-x) & x < 0 \end{cases}$

So $v \in L^p(\mathbb{R})$ with $\|v\|_{L^p(\mathbb{R})} = 2^{1/p} \|u'\|_{L^p(I)}$

<u>Need to show</u>: v weak derivative of u^* on $\mathbb{R}$.

Since $n=1$, $u \in C[0,\infty)$ (u has a cont. representative), and

$\forall x \geq 0$, $u(x) = u(0) + \int_0^x u'(y)\,dy = u(0) + \int_0^x v(y)\,dy$

If $x<0$, then $u^*(x) = u(-x) = \left(u(0) + \int_0^{-x} u'(y)\,dy\right)$

$$= u(0) + \int_0^x -u'(-y)\,dy$$

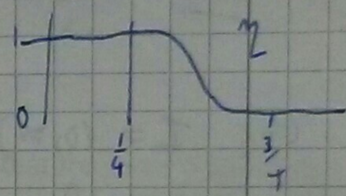
$$= u(0) + \int_0^x v(y)\,dy$$

Hence $\forall x\in\mathbb{R}$, $u^*(x) = u^*(0) + \int_0^x v(y)\,dy$

Hence (see proof of Satz 59), u^* weakly diff'ble on $\mathbb{R}$ with weak derivative v.

(b) Suppose $I=(0,1)$. Let $\eta\in C^1(\mathbb{R})$ s.t. $0\le\eta\le1$,

$$\eta(x) = \begin{cases}1 & x<\frac14\\ 0 & x>\frac34\end{cases}$$



For $u\in W^{1,p}(0,1)$, define

$$(Qu)(x) := \begin{cases} u(x) & 0<x<1 \\ 0 & x>1\end{cases}$$ extend by 0 to $(0,\infty)$ $= (\check{u})$

Claim: $\eta\check{u} \in W^{1,p}(0,\infty)$, $(\eta\check{u})' = \eta'\check{u} + \eta\check{u}'$.

Pf: Let $\varphi\in C_c^\infty(0,\infty)$. Then

$$\int_0^\infty (\eta\check{u})\varphi'\,dx = \int_0^1 \eta\check{u}\varphi'\,dx$$

$$= \int_0^1 u\left((\eta\varphi)' - \eta'\varphi\right)dx$$

$\eta\varphi\in C^1_c(0,1)$ defn of weak deriv holds even if test fns are C^1_c

$$= \int_0^1 -u'\eta\varphi - u\eta'\varphi\,dx = -\int_0^\infty(\check{u}'\eta - \check{u}\eta')\varphi$$

easy to verify $\eta\check{u}$, $(\eta\check{u})'$ in $L^p(0,\infty)$.

$\|\eta\check{u}\|_{L^p(\mathbb{R})} \le \|u\|_{L^p(I)}$ $\quad \|\eta Q(u)'\|_p \le \|\eta'\|_\infty \|u\|_{L^p(I)} + \|u'\|_{L^p(I)}$

(c) Let $u \in W^{1,p}(0,1)$.

Write $u = \eta u + (1-\eta) u$

~~Let v~~ Extend ηu to $W^{1,p}(0,\infty)$ as in part (b) and reflect to $\mathbb{R}$ as in part (a).

So $v_1 = P(\eta(Qu))$. $\|v_1\|_{L^p(\mathbb{R})} \le 2^{1/p} \|u\|_{L^p(I)}$

$\|v_1\|_{W^{1,p}(\mathbb{R})} \le C \|u\|_{W^{1,p}(I)}$ (C depends on $\|\eta'\|_{L^\infty}$)

In a similar way extend $(1-\eta)u$ to $(-\infty, 1)$ by 0 on $(-\infty, 0)$:

(note $(1-\eta)(x) = \begin{cases} 0 & x < 1/4 \\ 1 & x > 3/4 \end{cases}$)

Then ~~reflect~~ extend to $\mathbb{R}$ by reflection (about the point 1)

We get a function v_2 s.t.

$\|v_2\|_{L^p(\mathbb{R})} \le 2^{1/p} \|u\|_{L^p(I)}$

$\|v_2\|_{W^{1,p}(\mathbb{R})} \le C \|u\|_{W^{1,p}(I)}$.

Then $Pu := v_1 + v_2$ satisfies conditions of the Theorem $\square$

② ~~Lemma~~ Proposition: Let $u \in W^{1,p}(0,\infty)$, $1 \le p < \infty$

Then $\lim_{x \to \infty} u(x) = 0$.

Proof: From Satz 61, $\exists$ a sequence $(u_j) \subset C^1_c(\mathbb{R})$ such that $u_j|_{(0,\infty)} \to u$ in $W^{1,p}(0,\infty)$.

By Satz 62, ($\exists C > 0$ s.t. $\|v\|_{L^\infty(0,\infty)} \le C\|v\|_{1,p}$ $\forall v \in W^{1,1}(0,\infty)$)

$\|u_j - u\|_{L^\infty(0,\infty)} \le C\|u_j - u\|_{W^{1,p}(0,\infty)} \to 0$ as $j \to \infty$.

Hence claim holds: Let $\varepsilon > 0$. Then $\exists j_0 \in \mathbb{N}$ s.t.

$\|u_j - u\|_\infty < \varepsilon$ $\forall j \ge j_0$.

$(u_j)_{1 \le j \le j_0}$ all compactly supported functions: finitely many.

So $\bigcup_{j=1}^{j_0} \operatorname{supp} u_j$ is $\subseteq$ compact, $(-K, K)$, some K finite.

Then $\forall x > K$ $|u_j(x)| = 0$ ~~$\forall x$~~ $1 \le j \le j_0$
$< \varepsilon$ $j > j_0$.

□

③ Let $I \subseteq \mathbb{R}$, $1 \le p < \infty$ $u, v \in W^{1,p}(I)$.

Proposition: $uv \in W^{1,p}(I)$, with $(uv)' = u'v + uv'$.

Proof: $u, v \in L^\infty(I)$. Hence $uv \in L^\infty(I) \subseteq L^p(I)$. Also $u'v + uv' \in L^p(I)$.
Need to show this is weak deriv of uv.

Let $(u_j), (v_j) \subset C^\infty_c(\mathbb{R})$ with $u_j \to u$, $v_j \to v$ in $W^{1,p}(I)$.

Then (Satz 62) $u_j \to u$, $v_j \to v$ in $L^\infty(I)$ too. $u_j v_j \to uv$ in $L^p(I)$.

Also, $(u_j v_j)' = u_j' v_j + u_j v_j'$. ~~and~~ $\to u'v + uv'$ in $L^p(I)$

~~$\int_I |u_j' v_j + u_j v_j' - u'v + uv'|^p \le \|v -$~~

Why? Note $u'v \in L^p(I)$ $\quad (u' \in L^p(I), v \in L^\infty(I))$

and $\|u_j' v_j - u'v\|_p \leq \|u_j' v_j - u_j' v\|_p + \|u_j' v - u'v\|_p$

$$\leq \|v_j - v\|_\infty \underbrace{\|u_j'\|_p}_{\text{bdd as } u_j' \text{ converges in } L^p} + \|v\|_\infty \|u_j' - u'\|_p$$

Similarly $u_j v_j' \longrightarrow u v'$ in $L^p(I)$.

Hence, given $\varphi \in C_c^\infty(I)$,

$$\int_I uv\,\varphi' = \lim_{j\to\infty} \int_I u_j v_j\,\varphi'$$

(see pf that $W^{1,p}$ is a B.S.)

$$= \lim_{j\to\infty} -\int_I (u_j v_j)'\,\varphi$$

$$= -\int_I (u'v + uv')\,\varphi$$

Remarks: • This also holds for $W^{1,\infty}(I)$ (with slightly different proof - can't use density of $C_c^\infty(\mathbb{R})$)

• This does not in general hold when $n > 1$!!

Here we have made use of the cont. embedding $W^{1,p}(I) \hookrightarrow L^\infty(I)$

For $n > 1$ we need, for example $p > n$ and Ω has a C^1 boundary for this embedding.

Counterexample:

Last week, Q4: $B = B(0,1) \subset \mathbb{R}^2$ $u(x) = \frac{1}{|x|^s}$, $s \in (0,1)$

$u \in W^{1,p}(B)$ for $1 \leq p < \frac{2}{s+1}$ $(\notin L^\infty(B))$

Take $s = \frac{1}{2}$, $u(x) = v(x) = \frac{1}{|x|^{1/2}}$

Then $u, v \in W^{1,p}(B)$ for $1 \leq p < \frac{4}{3}$

But $(uv)(x) = \frac{1}{|x|} \notin W^{1,1}(B)$ (hence not in any $W^{1,p}(B)$)

④ Lax - Milgram Theorem:

Let H be a Hilbert Space and $a : H \times H \to \mathbb{R}$ a bilinear form s.t.

(i) $|a(u,v)| \leq C \|u\| \|v\|$ $\forall u, v \in H$ a is continuous

(ii) $a(u,u) \geq \varepsilon \|u\|^2$ $\forall u \in H$ a is coercive

Let $\varphi \in H^*$. Then $\exists$ unique $u_0 \in H$ s.t.

$a(u_0, v) = \varphi(v)$ $\forall v \in H$.

Consider ODE

$$(1) \begin{cases} -(pu')' + ru' + qu = f & \text{on } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

($p \geq \alpha$, $q \geq 1$, $r^2 < 4\alpha$, $f, q, r \in C[0,1]$, $p \in C^1[0,1]$, $\alpha > 0$.)

Look for weak solutions in Hilbert Space $W_0^{1,2}(0,1)$ $(= H_0^1(0,1))$

test with $v \in W_0^{1,2}(0,1)$. Weak formulate is:

Find $u \in W_0^{1,2}(0,1)$ s.t.

$$+\int_0^1 (pu')v' + ru'v + quv \, dx = \int_0^1 fv \, dx \quad \forall v \in W_0^{1,2}$$

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Weak formulation: If $u \in C^2[0,1]$ solves (1) then $\forall v \in C_c^\infty(0,1)$,

$$-\int_0^1 ((pu')' + ru' + qu)\, v = \int_0^1 fv$$

ie $$\int_0^1 pu'v' + ru'v + quv = \int_0^1 fv \qquad (2)$$

look for solution $u \in W_0^{1,2}(0,1)$ satisfying (2) $\forall v \in W_0^{1,2}(0,1)$

Define $a : H \times H \to \mathbb{R}$ by $\quad (H = W_0^{1,2}(0,1))$

$$a(u,v) = \int_0^1 pu'v' + ru'v + quv \, dx$$

- a billinear - clear (easy to check)

- a cont:

$$|a(u,v)| \le \|p\|_\infty \left|\int_0^1 u'v' \, dx\right| + \|r\|_\infty \left|\int u'v\right| + \|q\|_\infty \left|\int uv\right|$$

C-S in $L^2(0,1)$

$$\le \|p\|_\infty \|u'\|_{L^2} \|v'\|_{L^2} + \|r\|_\infty \|u'\|_{L^2} \|v\|_2 + \|q\|_\infty \|u\|_2 \|v\|_2$$

$$\le (\|p\|_\infty + \|r\|_\infty + \|q\|_\infty) \|u\|_{1,2} \|v\|_{1,2}$$

- a coercive

$$a(u,u) = \int_0^1 p(u')^2 + ru'u + q(u)^2$$

$p \ge \alpha$, $q \ge 1$

$$\ge \alpha \|u'\|_2^2 + \|u\|_2^2 + \int_0^1 ruu'$$

$$\ge \alpha \|u'\|_2^2 + \|u\|_2^2 - \|r\|_\infty \|u\|_2 \|u'\|_2$$

Need appro bnd on $\|r\|_\infty$ to guarantee this $\ge \varepsilon \|u\|_H^2 = \varepsilon(\|u\|_2^2 + \|u'\|_2^2)$ for some $\varepsilon > 0$.

$\forall y>0$: $x^2 + \alpha y^2 - \gamma xy \geq \varepsilon(x^2+y^2)$ find bds on γ.

$$(1-\varepsilon)x^2 + (\alpha-\varepsilon)y^2 - \gamma xy \geq 0.$$

$$\underbrace{(\sqrt{1-\varepsilon}\,x - \sqrt{\alpha-\varepsilon}\,y)^2}_{\geq 0} + (2\sqrt{1-\varepsilon}\sqrt{\alpha-\varepsilon} - \gamma)\,xy \geq 0$$

Need $\gamma < 2\sqrt{\alpha}$

ie $\gamma^2 < 4\alpha$ //.

Let $\varphi \in (W^{1,2}_0(0,1))^*$ be $\varphi(v) = \int_0^1 fv$

Then by Lax-Milgram $\exists$ unique solution $u_0 \in W^{1,2}_0(0,1)$ to (2).

<u>Show u_0 is a classical solution:</u> ie $u \in C^2[0,1]$, $u(0)=u(1)=0$.

Since $u_0 \in W^{1,2}_0(0,1)$, $u_0 \in C[0,1]$ with $u_0(0) = u_0(1) = 0$.

We have, for any $v \in W^{1,2}_0(0,1)$:

$$\int_0^1 pu'v' + ru'v + quv \, dx = \int_0^1 fv \, dx$$

So ~~$\int_0^1 (pv' + rv)u' = \int_0^1 (f-qu)v\,dx$~~

<u>Take v ~~smooth~~.</u>

$$\int_0^1 (pu')v' \, dx = \int_0^1 (f - ru' - qu)\, v \, dx \quad \text{true } \forall v \in C^\infty_c(0,1)$$

So pu' is weakly diff'ble with weak deriv

(3) $(pu')' = ru' + qu - f \in L^2(0,1)$. So $pu' \in W^{1,2}_0(0,1)$?.
(certainly)

So pu' is in $C[0,1]$, p is in $C^1[0,1]$. ~~Hence~~, $p \geq 1$.

~~u' must be $C[0,1]$~~. $u'(x) = $ ~~$p(x)u'(x)$~~ $(pu')(x) \cdot \frac{1}{p(x)} \in C[0,1]$

$\in C^1$, since $p \geq 1$

But then by (3)

$$(pu')' \in C[0,1].$$

So then $pu' \in C^1[0,1]$, so $u' \in C^1[0,1]$.

Hence $u \in C^2[0,1]$ //